Radon transform of Wheeler-De Witt equation and tomography of quantum states of the universe

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Abstract

The notion of standard positive probability distribution function (tomogram) which describes the quantum state of universe alternatively to wave function or to density matrix is introduced. Connection of the tomographic probability distribution with the Wigner function of the universe and with the star-product (deformation) quantization procedure is established.

Using the Radon transform the Wheeler-De Witt generic equation for the probability function is written in tomographic form. Some examples of the Wheeler-DeWitt equation in the minisuperspace are elaborated explicitly for a homogeneous isotropic cosmological models. Some interpretational aspects of the probability description of the quantum state are discussed.

1 Introduction

Recently in conventional quantum mechanics the Radon [1] transform of the von Neumann density operator [2][3] considered in the form of Wigner function [4] was recognized to give the tomographic probability (called tomographic map or tomogram) appropriate to reconstruct quantum states [5][6][7]. The slightly modified Radon transform of density matrix with additional scaling transform was suggested [8] in which the problem of the singularity of the Radon transform using only a rotation parameter was smoothed.

The possibility to implement the tomographic probability to define the quantum state in terms of conventional probability was suggested in [9]. It was pointed out that there exists a representation in quantum mechanics in which any quantum state can be described by a standard positive probability distribution. In [10] it was shown how superposition principle (quantum interference) is described using only positive probabilities. The properties of the tomographic map and its relation to the Heisenberg-Weyl group, SU(2)-group and to star-product quantization were discussed in [11].

It was understood [11] that the tomographic map is closely related to the star-product quantization procedure which provides Moyal equation for quantum evolution in the phase-space representation of quantum states [12]

On the other hand quantum cosmology uses as basic notion the wave function of the universe (it would be better to say the wave functional) which depends on the metric and the material fields [13]. This wave functional obeys the Wheeler-De Witt equation [14] which is a generalisation of the Schrödinger equation for the wave function. The properties of the Wheeler-De Witt description of the universe are the subject of an intensive discussion [15][16][17][18] due to the importance of this approach in quantum cosmology (see review in [19]).

Interpretation of the wave function of the universe contains the same problems as the interpretation of the wave function of finite quantum systems[20]. The notion of density matrix of the universe is also used to describe the state of the universe (see for example[17]). The Weyl-Wigner representation of the density matrix and the corresponding deformation quantization procedure was used within the context of cosmological problems in [21] and [22].

The general study of deformation quantization in quantum gravity is a highly non trivial procedure [21]. In our paper we use a particular deformation procedure related to the tomographic star product formalism [11].

The aim of our work is to introduce the tomographic probability function which describes the state of the universe and contains the same information on the universe state as the density matrix does. To reach this aim we apply the modified Radon transform for the density matrix of the universe, which is using the extension of the modified Radon transform of [8] for one degree of freedom. The Radon transform of the density matrix can be cast into the form of the transform of the wave-function [23]. We will discuss the extension of the functional Radon transform both in the form of the transform of the wave functional of the universe and in the form of the transform of the density matrix functional. Our procedure is an heuristic one and the rigorous mathematics of functional measures and of convergence of integral functionals needs further investigations, see for example [24] [25].

Our goal is to obtain the Wheeler-De Witt equation in tomographic (or probability) representation written for the tomogram of the universe. We consider also the simplest case of Wheeler-De Witt equation for the wave functional in which all the variety of metrics is reduced to the variety of radial time dependence. In this case the Wheeler-De Witt equation takes the form of Schrödinger-like equation for one degree of freedom. We will reobtain this equation in the form of the equation for tomographic probability.

The paper is organized as follows. In the next section 2 we review the star product procedure (or the deformation procedure) in a general form. In section 3 we present the tomographic approach in the phase space. In section 4 the functional Radon transform is discussed. In section 5 examples of tomographic representation of the Wheeler-DeWitt equation for one degree of freedom are given. Using the extension of the tomographic functional of a scalar field [26] the notion of probability functional of the universe is introduced in Appendix.

Also the generic Wheeler-DeWitt equation is written in the form of von Neumann like equation for density matrix functional and by means of the functional Radon transform it is rewritten in tomographic form in Appendix. Perspectives of the probability description of the universe state are discussed in the conclusions, section 6.

2 Star-product deformation quantization

In this section we review the quantization procedure. This approach uses a deformation procedure [21]. Another explanation of the procedure is in-

troducing and using a star-product of the operator symbols [27]. Below we follow the presentation of the star-product as given in [11]. Let us consider a Hilbert space $\mathcal{H}$ and a set of operators acting in the Hilbert space. The state of the universe can be associated either with a vector in the Hilbert space or with a density operator $\hat{\rho}$ which is a nonnegative Hermitian operator. Let us consider an operator $\hat{A}$. Let us suppose also that there exists a set of operators $\hat{U}(\vec{x})$ where $\vec{x} = (x_1, x_2..., x_N)$ such that the function (called the symbol of the operator $\hat{A}$)

$$f_{\hat{A}}(\vec{x}) = Tr(\hat{A}\hat{U}(\vec{x})) \tag{1}$$

defines the operator completely. It means that there exists a dual set of operators $\hat{\mathcal{D}}(\vec{x})$ such that one has the relation

$$\hat{A} = \int f_{\hat{A}}(\vec{x})\hat{\mathcal{D}}(\vec{x})d\vec{x}. \tag{2}$$

If there exist such operator families $\hat{U}(\vec{x})$ and $\hat{\mathcal{D}}(\vec{x})$, one can introduce the star-product of symbols defined by the relation

$$f_{\hat{A}\hat{B}}(\vec{x}) = f_{\hat{A}}(\vec{x}) * f_{\hat{B}}(\vec{x}) := Tr(\hat{A}\hat{B}\hat{U}(\vec{x})). \tag{3}$$

In view of the associativity of the operator product the star-product is also associative, i.e.

$$f_{\hat{A}}(\vec{x}) * (f_{\hat{B}}(\vec{x}) * f_{\hat{C}}(\vec{x})) = (f_{\hat{A}}(\vec{x}) * f_{\hat{B}}(\vec{x})) * f_{\hat{C}}(\vec{x}). \tag{4}$$

In our article we shall discuss two types of symbols associated with operators. The first type is called the Weyl symbol. For a state density operator Weyl symbol is the Wigner function. The Weyl symbol $W_{\hat{A}}(q,p)$ of an operator $\hat{A}$ is defined by the following families of operators

$$\hat{U}(\vec{x}) = \hat{U}(x_1, x_2) \tag{5}$$

and

$$\hat{\mathcal{D}}(\vec{x}) = \hat{\mathcal{D}}(x_1, x_2) \tag{6}$$

for which we assume $x_1 = q/\sqrt{2}$, $x_2 = p/\sqrt{2}$ (we consider a quantum system with one degree of freedom). Thus we introduce the two family of operators

$$\hat{U}(q,p) = 2\hat{\mathcal{D}}(\alpha)(-1)^{a^\dagger a}\hat{\mathcal{D}}(-\alpha), \quad \alpha = \frac{1}{\sqrt{2}}\left(q + ip\right), \tag{7}$$

$$\hat{\mathcal{D}}(q,p)=\frac{1}{\pi}\hat{\mathcal{D}}(\alpha)(-1)^{a^\dagger a}\hat{\mathcal{D}}(-\alpha). \tag{8}$$

Here the operators $a^\dagger$ and a are bosonic creation and annihilation operators

$$a=\frac{1}{\sqrt{2}}\left(\hat{q}+i\hat{p}\right). \tag{9}$$

In position representation the operators $\hat{q}$ and $\hat{p}$ are given by the standard relation

$$\hat{q}\psi(x)=x\psi(x), \qquad \hat{p}\psi(x)=-i\frac{\partial\psi}{\partial x}, \tag{10}$$

$\hbar=1$.

Note that the introduced operators have two aspects, one related to the linear transformations in the coordinate space, the other has to do with the unitary representations of the translation group.

In (7) and (8) the operator $\hat{\mathcal{D}}(\alpha)$, where α is the complex number defined in (7), is a unitary displacement operator

$$\hat{\mathcal{D}}(\alpha)=\exp(\alpha a^\dagger-\alpha^* a). \tag{11}$$

The operator $(-1)^{a^\dagger a}$ is the parity operator. Thus the Weyl symbol of the operator $\hat{A}$ is defined by the relation

$$W_{\hat{A}}=2Tr(\hat{A}\hat{\mathcal{D}}(\alpha)(-1)^{a^\dagger a}\hat{\mathcal{D}}(-\alpha). \tag{12}$$

If $\hat{A}$ is a density operator $\hat{\rho}$ defining a state of a quantum system (the state of the universe) the relation (12) provides the Wigner function of the state. In our paper we will study another symbol $\mathcal{W}_{\hat{A}}(X,\mu,\nu)$ of the operator $\hat{A}$ called the tomographic symbol.

The tomographic symbol is defined by means of the pair of families of operators $\hat{U}(\vec{x})$ and $\hat{\mathcal{D}}(\vec{x})$ where $\vec{x}=(X,\mu,\nu)$ and X, μ, ν are real numbers. The operators are given by the formulae

$$\hat{U}(X,\mu,\nu)=\delta(X-\mu\hat{q}-\nu\hat{p}) \tag{13}$$

$$\hat{\mathcal{D}}(X,\mu,\nu)=\frac{1}{2\pi}e^{iX}e^{i(\mu\hat{q}+\nu\hat{p})}. \tag{14}$$

Thus the symbol of an operator $\hat{A}$, called the tomogram, is given by the relation

$$\mathcal{W}_{\hat{A}}(X,\mu,\nu)=Tr(\hat{A}\delta(X-\mu\hat{q}-\nu\hat{p})). \tag{15}$$

The inverse relation reads

$$\hat{A} = \frac{1}{2\pi} \int \mathcal{W}_{\hat{A}}(X, \mu, \nu) e^{iX + i\mu\hat{q} + i\nu\hat{p}} dX d\mu d\nu. \tag{16}$$

The star-product of two tomograms is defined using the kernel

$$\begin{aligned} \mathcal{W}_{\hat{A}}(X, \mu, \nu) * \mathcal{W}_{\hat{B}}(X, \mu, \nu) = & \int dX_1 d\mu_1 d\nu_1 dX_2 d\mu_2 d\nu_2 \mathcal{W}(X_1, \mu_1, \nu_1) \\ & \times \mathcal{W}(X_2, \mu_2, \nu_2) K(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, X, \mu, \nu) \end{aligned} \tag{17}$$

Here the kernel is given by

$$K(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, X, \mu, \nu) = Tr \left\{ \hat{\mathcal{D}}(X_1, \mu_1, \nu_1) \hat{\mathcal{D}}(X_2, \mu_2, \nu_2) \hat{U}(X, \mu, \nu) \right\}. \tag{18}$$

This trace can be calculated and the tomographic kernel reads [11]

$$\begin{aligned} K(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, X, \mu, \nu) = & \frac{\delta(\mu(\nu_1 + \nu_2) - \nu(\mu_1 + \mu_2))}{4\pi} \\ \times \exp \left(\frac{i}{2} \left\{ (\nu_1 \mu_2 - \nu_2 \mu_1) + 2X_1 + 2X_2 - \left[\frac{\nu_1 + \nu_2}{\nu} + \frac{\mu_1 + \mu_2}{\mu} \right] X \right\} \right) & \end{aligned} \tag{19}$$

For the multimode case as well for the infinite dimensional (functional) case the generalization is in principle straightforward. In the case of both Weyl symbols and tomographic symbols, one simply provides an index (either a discrete or a continuous one) to the involved ingredients q, p and X, μ, ν. In the infinite dimensional case the Wigner symbol and the tomogram of the operator $\hat{A}$ become functionals. Correspondingly one modifies the integration measures by the standard procedure.

3 Radon transform of Wigner function and Fractional Fourier transform of wave function

In this section we consider the relations of a tomographic symbol with the Radon transform of the Weyl symbol. In order to write the Wheeler-DeWitt equation in tomographic form we review some properties of the modified Radon transform of Wigner function [4]. The Wigner function is the Weyl symbol of the von Neumann density matrix [2].

The Wigner function is expressed in terms of density matrix of the universe in the form ($\hbar = 1$)

$$W(q,p) = \int \rho\left(q + \frac{u}{2}, q - \frac{u}{2}\right) e^{-ipu} du \tag{20}$$

The inverse transform reads

$$\rho(x,x') = \frac{1}{2\pi} \int W\left(\frac{x+x'}{2}, p\right) e^{ip(x-x')} dp. \tag{21}$$

The Radon transform of the Wigner function in the modified form is the integral transform of the form

$$\mathcal{W}(X,\mu,\nu) = \int W(q,p) e^{ik(X-\mu q-\nu p)} \frac{dk dq dp}{(2\pi)^2} \tag{22}$$

Here X, μ ν are real numbers. The Wigner function can be found using the inverse Radon relation

$$W(q,p) = \frac{1}{2\pi} \int e^{i(X-\mu q-\nu p)} \mathcal{W}(X,\mu,\nu) dX d\mu d\nu. \tag{23}$$

The standard Radon transform is obtained from the two above by taking $\mu = \cos\varphi$, $\nu = \sin\varphi$.

One can see that the tomographic symbol of density matrix is given as a marginal distribution since

$$\mathcal{W}(X,\mu,\nu) = \int W(q,p) \delta(X - \mu q - \nu p) \frac{dq dp}{2\pi} \tag{24}$$

It is clear that

$$\int \mathcal{W}(X,\mu,\nu) dX = 1, \tag{25}$$

since the Wigner function is normalized

$$\int W(q,p) \frac{dq dp}{2\pi} = 1 \tag{26}$$

for normalized wave functions.

The formulae (24) – (26) are valid for arbitrary density matrices, both for pure and mixed states. For pure states of the universe, the tomographic symbol can be expressed directly in terms of the wave function of the universe using [23],

$$\mathcal{W}(X,\mu,\nu)=\frac{1}{2\pi|\nu|}\left|\int\psi(y)e^{\frac{i\mu}{2\nu}y^2-\frac{iX}{\nu}y}dy\right|^2. \tag{27}$$

The inverse transform provides the wave function due to the relation

$$\psi(y)\psi^*(y')=\frac{1}{2\pi}\int\mathcal{W}(X,\mu,y-y')e^{i\left(X-\mu\frac{y+y'}{2}\right)}dX\,d\mu \tag{28}$$

which for the mixed states reads

$$\rho(x,x')=\frac{1}{2\pi}\int\mathcal{W}(X,\mu,x-x')e^{i\left(X-\mu\frac{x+x'}{2}\right)}dX\,d\mu. \tag{29}$$

In fact, both the Weyl symbol and the tomographic symbol of density matrix can be cast into framework of the theory of the maps of operators acting in Hilbert space of states onto functions, the pointwise product of functions being replaced by the star product[11].

The formula relating the tomographic symbol with the wave function contains the integral

$$I=\left|\int\psi(y)e^{\frac{i\mu}{2\nu}y^2-\frac{iX}{\nu}y}dy\right| \tag{30}$$

In case of $\mu=0$, $\nu=1$ this integral is a conventional Fourier transform of the wave function. For generic μ, ν the integral is identical to the modulus of Fractional Fourier transform of the wave function [23][28]. Thus, the Radon transform of the Wigner function in the case of pure states is related to the Fractional Fourier transform of the wave function. From the linear integral relations for the density matrix $\rho(x,x')$ and the tomographic probabilities $\mathcal{W}(X,\mu,\nu)$ follow the identities (see e.g. [29])

$$\begin{aligned}
&\rho(x,x')\to\mathcal{W}(X,\mu,\nu),\\
&x\to-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}+\frac{i}{2}\nu\frac{\partial}{\partial X},\\
&x'\to-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}-\frac{i}{2}\nu\frac{\partial}{\partial X};\\
&\frac{\partial}{\partial x}\to\frac{1}{2}\mu\frac{\partial}{\partial X}-i\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\nu},
\end{aligned} \tag{31}$$

$$\frac{\partial}{\partial x'} \to \frac{1}{2}\mu\frac{\partial}{\partial X} + i\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \nu}. \tag{32}$$

The physical meaning of the random variable X and the two real parameters μ and ν is the following one [9] [11]. The variable X is the position of a quantum particle. But this position is considered in the specific rotated and scaled reference frame in phase-space. The reference frame is labeled by two parameters $\mu = \exp(\lambda)\cos\theta$, $\nu = \exp(\lambda)\sin\theta$. The angle θ is the rotation angle and λ is the scaling factor. One has to point out that the tomographic map can be applied to arbitrary functions which satisfy equations of different types, like elliptic-type and like wave equation of Klein-Gordon type, see e.g. [30]

4 Functional Radon transform and the Wheeler-DeWitt equation

In order to write the Wheeler-DeWitt generic equation for the wave functional of the universe, one needs to present the generalization of the formulae of the previous sections to the case of functionals which we identify with the functions of an infinite number of variables $\psi(x_1, x_2, \ldots)$. We can write these functions in the form $\psi(x(k))$. Replacing $k \to \tau$ one sees that the functional depends on the function $\psi(x(\tau))$ where τ is a continuous variable. One can extend the notion of functional considering the parameter τ to be a vector with several continuous components (e.g., $\tau = (\tau_1, \tau_2, \tau_3, \tau_0)$, like space-time variables). Also the number of functions can be extended such that

$$x(\tau) \to (x_1(\tau), \ldots x_k(\tau), \ldots x_N(\tau)).$$

The index $1, 2, \ldots N$ can be considered as counting some set of indices like $(a, b, c, d) \equiv k$ where the numbers a, b, c, d are tensor indices. In this sense we omit all the indices and will treat the functional $\psi(x(\tau))$ in the discussed generic sense, considering x as a vector and τ as a vector. One knows that for the notions of derivative for functionals $\delta\psi(x(\tau))/\delta(x(\tau'))$ can be introduced simply as a generalization of the partial derivative of a function of several variables $\partial\psi(x_i)/\partial x_k$. Given an equation for the density matrix functional one can get the corresponding equations for the tomographic probability functional. To this aim one has to use the replacements (31) and (32),

modified for an infinite number of variables, in the equation for the density matrix.

Thus if one has the wave functional $\psi(x(\tau))$, the corresponding density matrix functional is

$$\rho(x(\tau), x'(\tau')) = \psi(x(\tau))\psi^*(x'(\tau)). \tag{33}$$

One can introduce the Wigner function of the universe defining it for a pure state as

$$W(q(\tau), (p(\tau)) = \int \psi\left(q(\tau) + \frac{u(\tau)}{2}\right)\psi^*\left(q(\tau) - \frac{u(\tau)}{2}\right) \times\ e^{-i\int u(\tau)p(\tau)d\tau}\mathcal{D}[u(\tau)] \tag{34}$$

where $\mathcal{D}[u(\tau)]$ is the measure in the Fourier functional integral. The tomographic probability becomes also the functional $\mathcal{W}(X(\tau), \mu(\tau), \nu(\tau))$ which is given in terms of the Wigner functional of the universe as

$$\mathcal{W}(X(\tau), \mu(\tau), \nu(\tau)) = \int W\left(q(\tau), p(\tau)\right)\delta[X(\tau) - \mu(\tau)q(\tau) - \nu(\tau)q(\tau)] \times \mathcal{D}(q(\tau), p(\tau)) \tag{35}$$

Thus the tomographic probability functional is given by the above formula which is the functional Radon transform of the Wigner functional.

The universe in a model of quantum cosmology is described by a wave functional which depends on the spatial metric. This wave functional obeys the Wheeler-DeWitt equation of the form [14]

$$\left[-G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta_{kl}} - \ {}^3R(h)h^{1/2} + 2\Lambda h^{1/2}\right]\Psi[h_{ij}] = 0 \tag{36}$$

where h_{ij} is the spatial metric, G_{ijkl} is the metric on the space of three geometries (superspace)

$$G_{ijkl} = \frac{1}{2}h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}) \tag{37}$$

and ${}^3R(h)$ is the scalar curvature of the intrinsic geometry of the three-surface, Λ is the cosmological constant. It means that the density matrix functional and the analog of the Wigner function in the form of a functional

can be introduced as well as a tomogram functional of the the quantum state of the universe. Below in Appendix we will write this equation in tomographic form equation. But to make transition to tomographic representation clearer we discuss first simple cosmological models.

In the following we shall consider different examples of a homogeneous and isotropic universe. Even if they can be referred to the same geometry, these examples are not equivalent from the quantization point of view. As a matter of fact, it is well-known that the canonical formulation of gravity leads to the breaking of the covariance of the theory with respect of the group of four dimensional diffeomorphisms. Therefore any change of coordinates does not necessarily lead to a canonical transformation in the Hamiltonian formulation of General Relativity.

The evolution of the spatial metric is considered in the context of the space of (spatial) metrics, i.e. the superspace. When a homogeneous model is considered, the spatial metric is parameterized by functions of time and a model equivalent to a classical particle results. In this case the evolution is considered in a restricted version of the superspace, i.e. the so-called minisuperspace. In the case of a Friedmann-Lemaitre-Robertson-Walker the minisuperspace is a described by particle in one dimension. The presence of fields like a scalar field would eventually extend the minisuperspace dimensions.

There exist several elaborated examples of minisuperspaces used in quantum cosmology, below we consider some of these examples.

5 Some examples of the Wheeler-DeWitt equations

5.1 Homogeneous and isotropic universe with cosmological constant and no material source

In our first example we consider the model in which the metric dependence is reduced to dependence only on the expansion factor of the universe. This is a one dimensional Wheeler-DeWitt equation for a FLRW universe of the form

$$\frac{1}{2}\left\{\frac{1}{a^p}\frac{d}{da}a^p\frac{d}{da}-a^2+\Lambda a^4\right\}\psi(a)=0 \tag{38}$$

Here $0 \leq a < +\infty$, is in the classical theory the expansion factor and p is an index introduced to take into account the ambiguity of operator ordering. The Radon transform discussed in previous sections makes sense only for variables that take values from $-\infty$ to $+\infty$, so we make the change of variables $a = \exp x$ and the Wheeler-DeWitt equation becomes

$$\frac{1}{2}\left\{\exp(-2x)\frac{d^2}{dx^2} + (p-1)\exp(-2x)\frac{d}{dx} - 2U(x)\right\}\Psi(x) = 0 \qquad (39)$$

where $U(x) = (\exp(2x) - \Lambda\exp(4x))/2$. This equation can be written also in the form

$$\frac{1}{2}\left\{\exp(-2x')\frac{d^2}{dx'^2} + (p-1)\exp(-2x')\frac{d}{dx'} - 2U(x')\right\}\Psi^*(x') = 0. \qquad (40)$$

Multiplying the two equations respectively by $\Psi^*(x')$ and by $\Psi(x)$, and taking the difference, we finally obtain the equation for the density matrix $\rho(x,x') = \Psi(x)\Psi^*(x')$

$$\frac{1}{2}\left\{\left[\exp(-2x)\left(\frac{d}{dx}\right)^2 - \exp(-2x')\left(\frac{d}{dx'}\right)^2\right]\right.$$

$$\left.+\frac{1}{2}(p-1)\left[\exp(-2x)\frac{d}{dx} - \exp(-2x')\frac{d}{dx'}\right] - (U(x) - U(x'))\right\}\rho(x,x') = 0 \quad (41)$$

Using equations (31) and (32) we get the equation

$$\left\{\mathrm{Im}\left[\exp\left[2\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu} + i\nu\frac{\partial}{\partial X}\right]\left(\frac{1}{2}\mu\frac{\partial}{\partial X} - i\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\nu}\right)^2\right]\right.$$

$$+(p-1)\mathrm{Im}\left[\exp\left(2\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu} + i\nu\frac{\partial}{\partial X}\right)\left(\frac{1}{2}\mu\frac{\partial}{\partial X} - i\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\nu}\right)\right]$$

$$-2\mathrm{Im}\left[\exp\left(-2\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu} + i\nu\frac{\partial}{\partial X}\right) - \Lambda\exp\left(-4\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}\right.\right.$$

$$\left.\left.\left.+ 2i\nu\frac{\partial}{\partial X}\right)\right]\right\}\mathcal{W}(X,\mu,\nu) = 0, \qquad (42)$$

this equation is the tomographic form of the equation (39). There is no exact solution of equation (39), but for very large a the solution has the form (see [16])

$$\psi(a) \sim \cos \frac{Ha^3}{3} \tag{43}$$

The expression for the tomogram in this case is

$$\mathcal{W}(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \cos \frac{Hy^3}{3} e^{i\mu y^2/2\nu} e^{-iXy/\nu} dy \right|^2 . \tag{44}$$

5.2 Homogeneous and isotropic universe with a different metric

In [18][31] a (closed) homogeneous and isotropic universe is considered, but where the metric is expressed in a coordinate system such that it takes the form

$$ds^2 = -\frac{N^2(t)}{q(t)} dt^2 + q(t) d\Omega_3^2. \tag{45}$$

In this case the Wheeler-DeWitt equation assumes the form

$$\frac{1}{2} \left(4 \frac{d^2}{dq^2} + \lambda q - 1 \right) \psi(q) = 0, \tag{46}$$

where λ is a parameter related to the cosmological constant Λ and the gravitational constant G by the relation $\lambda = 2\, G\, \Lambda / 9\pi$ (see [31]).

This equation can be expressed in the following form

$$\frac{d^2 \psi(\xi)}{d\xi^2} + \xi \psi(\xi) = 0 \tag{47}$$

where

$$\xi = \left(\frac{\lambda}{2} \right)^{1/3} \left(q - \frac{1}{\lambda} \right). \tag{48}$$

The solution of equation (47) is

$$\psi(q) = A\Phi(-\xi) = A\Phi \left(-\frac{\lambda}{2} \right)^{1/3} \left(q - \frac{1}{\lambda} \right) \tag{49}$$

where $\Phi(x)$ is the Airy function

$$\Phi(x) = \int_0^\infty \cos\left(\frac{u^3}{3} + u\,x\right) du, \tag{50}$$

A is a normalization constant. The equation for the tomogram is the same equation for the tomogram of an electric charge moving in a constant homogeneous electric field and it reads [32]

$$-\mu\frac{\partial\mathcal{W}}{\partial\nu} + F\nu\frac{\partial\mathcal{W}}{\partial X} = 0 \tag{51}$$

The corresponding expression for the tomogram is

$$\mathcal{W}(X,\mu,\nu) = \frac{A^2}{2\pi|\nu|}\left|\Phi\left(-\left(\frac{1}{2\lambda^2}\right)^{1/3} + \left(\frac{\lambda}{2}\right)^{1/3}\frac{X}{\mu} - \left(\frac{\lambda}{2}\right)^{2/3}\frac{\nu^2}{\mu^2}\right)\right|^2 \tag{52}$$

5.3 The "harmonic oscillator" case

Hartle and Hawking showed [16] that the Wheeler-DeWitt equation for a homogeneous and isotropic metric

$$ds^2 = \sigma^2(N^2 d\tau^2 + a^2(\tau)d\Omega_3^2), \tag{53}$$

with a conformally invariant field φ and zero cosmological constant, reduces to the equation of a harmonic oscillator

$$\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \omega_1^2 x^2 - \frac{\partial^2}{\partial y^2} + \omega_2^2 y^2\right)\psi(x,a) = 0 \tag{54}$$

with $x = a$, $y = \phi\,a$ and $\omega_1 = \omega_2 = 1$. But Gousheh and Sepangi [33] pointed out that the equation (54) holds also for other cosmological models. For example by taking a scalar field ϕ with potential

$$V(\phi) = \lambda + \frac{m^2}{2\alpha^2}\sinh^2(\alpha\phi) + \frac{b}{2\alpha^2}\sinh(2\alpha\phi) \tag{55}$$

one can obtain, by suitable changes of coordinates, equation (54). The same equation can be derived in a Kaluza-Klein cosmology with negative cosmological constant and metric

$$ds^2 = -dt^2 + a^2(t)\frac{\delta_{ij}dx^i dx^j}{(1+\frac{kr^2}{4})} + A^2(t)d\varrho^2 \tag{56}$$

where $A(t)$ is the radius of the compactified dimension.

A solution of equation (54) can be obtained by separation of variables [33]

$$\psi_{n_1n_2}(x,y)=\alpha_{n_1}(x)\beta_{n_2}(y)\qquad n_1,n_2=\ 0,1,2,\dots \tag{57}$$

where both the families of functions $\alpha_n(x)$ and $\beta_n(y)$ are expressed by

$$\alpha_n(x)=\left(\frac{1}{\pi}\right)^{1/4}\frac{H_n(x)}{\sqrt{2^n n!}}e^{-\,x^2/\,2} \tag{58}$$

and

$$\beta_n(y)=\left(\frac{1}{\pi}\right)^{1/4}\frac{H_n(y)}{\sqrt{2^n n!}}e^{-\,y^2/\,2}. \tag{59}$$

One can obtain by the described method the corresponding equation for the tomogram and it reads

$$-\,\mu_1\frac{\partial\mathcal{W}}{\partial\nu_1}+\nu_1\frac{\partial\mathcal{W}}{\partial\mu_1}+\mu_2\frac{\partial\mathcal{W}}{\partial\nu_2}-\nu_2\frac{\partial\mathcal{W}}{\partial\mu_2}=0 \tag{60}$$

The corresponding solution can be found by applying equation (27) to (57) and we obtain the tomographic symbol

$$\mathcal{W}_{n_1n_2}(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2)$$

$$=\frac{1}{(2\pi)^2|\nu_1\nu_2|}\left|\int\psi_{n_1n_2}(x,y)e^{\frac{i\mu_1x^2}{2\nu_1}}e^{\frac{i\mu_2y^2}{2\nu_2}}e^{-i\frac{xX_1}{\nu_1}}e^{-i\frac{yX_2}{\nu_2}}dx\,dy\right|^2$$

$$=\frac{1}{\pi}\frac{1}{2^{n_1+n_2}}\frac{1}{n_1!n_2!}\frac{e^{\frac{-X_1^2}{\mu_1^2+\nu_1^2}}\,e^{\frac{-X_2^2}{\mu_2^2+\nu_2^2}}}{\sqrt{(\mu_1^2+\nu_1^2)(\mu_2^2+\nu_2^2)}}H_{n_1}^2\left(\frac{X_1}{\sqrt{(\mu_1^2+\nu_1^2)}}\right)H_{n_2}^2\left(\frac{X_2}{\sqrt{(\mu_2^2+\nu_2^2)}}\right). \tag{61}$$

With the solutions found above, we can describe the tomogram for an entangled state of the universe. Entangled systems were already considered in the context of General Relativity by Basini et al. [34]. For instance, let us consider the combination which is the entangled superposition state of the universe in the model under study

$$\frac{1}{\sqrt{2}}\psi_{12}+\psi_{21}=\frac{1}{\sqrt{\pi}}(ye^{-\frac{x^2}{2}}e^{-\frac{y^2}{2}}+xe^{-\frac{x^2}{2}}e^{-\frac{y^2}{2}}); \tag{62}$$

the corresponding tomogram is

$$\mathcal{W}_{12}^{\text{entangled}}(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2)$$

$$= \frac{1}{2(2\pi)^2|\nu_1\nu_2|} \left| \int (\psi_{12}(x,y) + \psi_{21}(x,y)) e^{\frac{i\mu_1 x^2}{2\nu_1}} e^{\frac{i\mu_2 y^2}{2\nu_2}} e^{-i\frac{xX_1}{\nu_1}} e^{-i\frac{yX_2}{\nu_2}} \right|^2$$

$$\begin{aligned} &= \frac{1}{(2\pi)^2|\nu_1\nu_2|} \left| 2\sqrt{\pi} \left(\frac{X_1(\mu_1 - i\nu_1))}{(\mu_1^2+\nu_1^2)} + \frac{X_2(\mu_2 - i\nu_2))}{(\mu_2^2+\nu_2^2)} \right) \right. \\ &\quad \left. \times \sqrt{\frac{1+i\mu_1/\nu_1}{1+\mu_1^2/\nu_1^2} \cdot \frac{1+i\mu_2/\nu_2}{1+\mu_2^2/\nu_2^2}}\; e^{-\frac{X_1^2(\nu_1+i\mu_1)}{2(\mu_1^2+\nu_1^2)}}\; e^{-\frac{X_2^2(\nu_2+i\mu_2)}{2(\mu_2^2+\nu_2^2)}} \right|^2 \end{aligned} \quad (63)$$

$$\begin{aligned} &= \frac{1}{\pi} \left(\frac{X_1^2}{(\mu_1^2+\nu_1^2)} + \frac{X_2^2}{(\mu_2^2+\nu_2^2)} + \frac{2X_1X_2(\mu_1\mu_2+\nu_1\nu_2)}{(\mu_1^2+\nu_1^2)(\mu_2^2+\nu_2^2)} \right) \\ &\quad \times \sqrt{\frac{1}{\nu_1^2+\mu_1^2} \cdot \frac{1}{\nu_2^2+\mu_2^2}}\; e^{-\frac{X_1^2}{(\mu_1^2+\nu_1^2)}}\; e^{-\frac{X_2^2}{(\mu_2^2+\nu_2^2)}}. \end{aligned} \quad (64)$$

This tomogram is the positive joint probability distribution of two random variables X_1 and X_2 and it completely determines the quantum state of the universe in the considered model.

6 Conclusions

To conclude we summarize the main results of our work. In the framework of quantum gravity we applied the recently introduced in quantum mechanics and quantum optics method of association with quantum states the probability distributions and in in view of this we managed to describe the states of the universe by standard positive probability distributions (tomograms of the universe states). We found the connection of this approach with star product (deformation) quantization. The conventional Wheeler-DeWitt for the wave function of the universe is presented in the form of a stochastic equation for the standard positive probability distributions. The wave function of the universe and its density matrix can be reconstructed in terms of the introduced tomographic probability distribution of the universe. Some

example of Friedmann-Lemaitre-Robertson-Walker minisuperspaces were explicitly studied and the tomograms of the corresponding universe states were showed including an entangled state. The description of an universe quantum state by standard positive probability distributions provides some new aspects to the problem of the connection with the classical description of pure universe states .

It is worthy to study these new aspects considering the classical limit of the quantum equations in the tomographic representation. One has to point out that for studying the classical limit one needs to take into account the decoherence phenomena which destroy the quantum coherence of the universe states. The classical limit of a quantum mechanical problem (kicked rotators) was discussed in tomographic representation in [35].

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7 Appendix

In the Appendix we review the details of the Radon transform approach to the Schrödinger equation and the Von Neumann equation. To do this we describe how the Schrödinger equation for the wave function induces the Von Neumann equation for the density matrix. After this the tomographic transform provides the equation for the tomogram of a quantum state.

The Schrödinger evolution equation for a system with one degree of freedom reads ($m = 1$)

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} + U(x)\psi(x,t); \tag{65}$$

$\hbar = 1$.

The Schrödinger equation energy-level equation reads

$$-\frac{1}{2}\frac{\partial^2\psi_E(x,t)}{\partial x^2} + U(x)\psi_E(x,t) = E\psi_E(x,t) \tag{66}$$

For the density matrix

$$\varrho(x,x',t) = \psi(x,t)\psi^*(x',t) \tag{67}$$

the von Neumann evolution equation can be obtained from equation (65) and it has the form

$$i\frac{\partial\varrho(x,x',t)}{\partial t} = -\frac{1}{2}\left[\frac{\partial^2\varrho(x,x',t)}{\partial x^2} - \frac{\partial^2\varrho(x,x',t)}{\partial x'^2}\right] + \left(U(x) - U(x')\right)\varrho(x,x',t) \tag{68}$$

Using the relations (27) and (29) one can see that the evolution equation for the tomogram of the quantum state can be obtained from the evolution equation for the density matrix (68) using the replacements

$$\varrho(x,x',t) \to \mathcal{W}(X,\mu,\nu,t), \tag{69}$$

(31) and (32). Thus the evolution equation for the quantum state tomogram has the form [9]

$$\frac{\partial\mathcal{W}}{\partial t} - \mu\frac{\partial\mathcal{W}}{\partial\nu} + \left[U(\tilde{q}) - U(\tilde{q}^*)\right]\mathcal{W} = 0 \tag{70}$$

where the argument of the potential is the operator

$$\tilde{q} = -\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu} + i\frac{\nu}{2}\frac{\partial}{\partial X}. \tag{71}$$

Here the operator $(\partial/\partial X)^{-1}$ is defined by the action onto the Fourier component $\tilde{f}(k)$ of a function $f(x)$

$$f(x) = \int\tilde{f}(k)e^{ikx}dk \tag{72}$$

due to the prescription

$$\left(\frac{\partial}{\partial X}\right)^{-1} f(x) = \int\frac{\tilde{f}(k)}{ik}e^{ikx}dk \tag{73}$$

The evolution equation for the tomogram (70) is the tomographic map of the Moyal equation [12] for the Wigner function $W(q,p,t)$

$$\frac{\partial W}{\partial t} + p\frac{\partial W}{\partial q} + \left[U(\tilde{\tilde{q}}) - U(\tilde{\tilde{q}}^*)\right]W = 0 \tag{74}$$

where the argument of the potential is the operator

$$\tilde{\tilde{q}} = q + \frac{i}{2}\frac{\partial}{\partial p}. \tag{75}$$

Thus introducing the functional

$$\rho(x,x') = \psi(x)\psi^*(x') \tag{76}$$

we get the Wheeler-DeWitt equation for the tomogram of the universe by means of the replacements

$$x \rightarrow -\left(\frac{\delta}{\delta X}\right)^{-1}\frac{\delta}{\delta\mu} + \frac{i}{2}\nu\frac{\delta}{\delta X}, \tag{77}$$

$$x' \rightarrow -\left(\frac{\delta}{\delta X}\right)^{-1}\frac{\delta}{\delta\mu} - \frac{i}{2}\nu\frac{\delta}{\delta X} \tag{78}$$

$$\frac{\delta}{\delta x} \rightarrow \frac{1}{2}\mu\frac{\delta}{\delta X} - i\left(\frac{\delta}{\delta X}\right)^{-1}\frac{\delta}{\delta\nu} \tag{79}$$

$$\frac{\delta}{\delta x'} \rightarrow \frac{1}{2}\mu\frac{\delta}{\delta X} + i\left(\frac{\delta}{\delta X}\right)^{-1}\frac{\delta}{\delta\nu} \tag{80}$$

which should be done in analogy with the von Neumann equation for the density of the universe [14]

$$\begin{aligned}&\left[-F_{\alpha\beta}\frac{\delta^2}{\delta x_\alpha \delta x_\beta} - 3R(x)S(x) + 2\Lambda S(x)\right.\\ &\left.-F_{\alpha\beta}(x')\frac{\delta^2}{\delta x'_\alpha \delta x'_\beta} - 3R(x')S(x') + 2\Lambda S(x')\right]\varrho(x,x') = 0\end{aligned} \tag{81}$$

Here

$$F_{\alpha\beta}(x) = -G_{ijkl}, \tag{82}$$

$$S(x) = h^{\frac{1}{2}}, \tag{83}$$

$$R(x) = {}^3R(h). \tag{84}$$

We take into account that the wave function of the universe is a real function. Making in (81) the replacement

$$\varrho(x,x') \rightarrow \mathcal{W}(X,\mu,\nu)$$

and using equations (77)–(80), we get the tomographic form of the Wheeler-DeWitt equation.

The considered cosmological models are particular cases of this general procedure.

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